

IMPROPER INTEGRALS

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IMPROPER INTEGRALS

Up to now we have studied integrals of the form

$$\int_a^b f(x)dx,$$

where f is a continuous function defined on the closed and bounded interval $[a, b]$.

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Up to now we have studied integrals of the form

$$\int_a^b f(x)dx,$$

where f is a continuous function defined on the closed and bounded interval $[a, b]$. Improper integrals are integrals in which one or both of these conditions are not met, i.e.

1) The interval of integration is not bounded $(-\infty, a]$, $[a, \infty)$, $(-\infty, \infty)$

$$\int_1^{\infty} \frac{1}{x^2} dx, \quad \int_{-\infty}^0 \frac{1}{1+x^3} dx, \quad \int_{-\infty}^{\infty} \frac{\sin x}{1+x^4} dx.$$

2) The integrand is discontinuous at $x = c$ where $c \in [a, b]$

$$\int_1^2 \frac{1}{\sqrt{2-x}} dx, \quad \int_0^2 \frac{1}{x-1} dx.$$

IMPROPER INTEGRALS

In this section, we extend the concept of a definite integral to the cases where:

- (i) The interval of integration is infinite;
- (ii) f has an infinite discontinuity in $[a, b]$.

- In either case, the integral is called an **improper integral**.

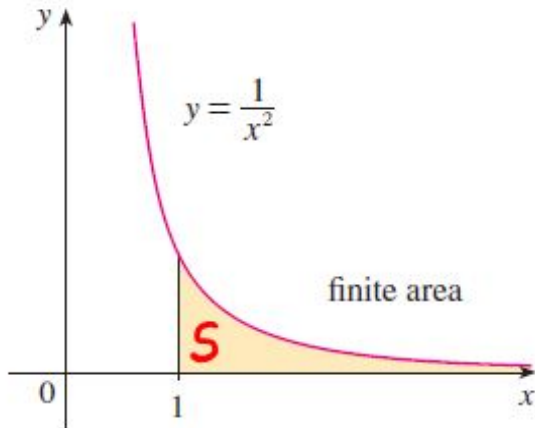
Integrals satisfying (i) are called **improper integral of type I**;

Integrals satisfying (ii) are called **improper integral of type II**.

Improper integrals of type I: Infinite limits of integration

Problem:

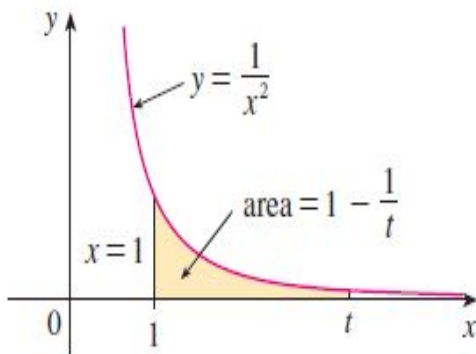
Find the area of the region S lying under the curve $y = 1/x^2$ and above the x -axis to the right of $x = 1$.



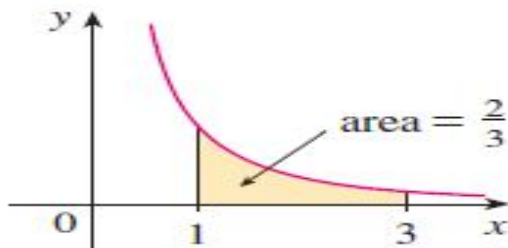
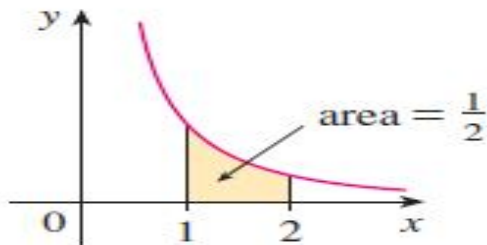
Improper integrals of type I: Infinite limits of integration

The area of the region bounded by the curve $y = \frac{1}{x^2}$, the x-axis and the straight lines $x = 1$ and $x = t$ ($t > 1$) is

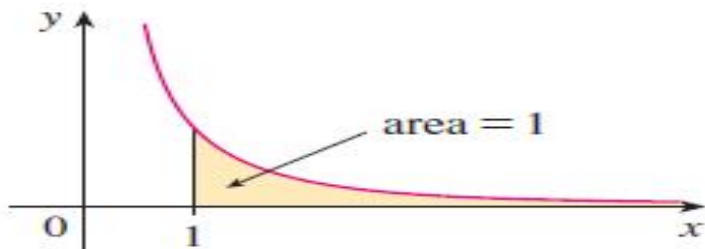
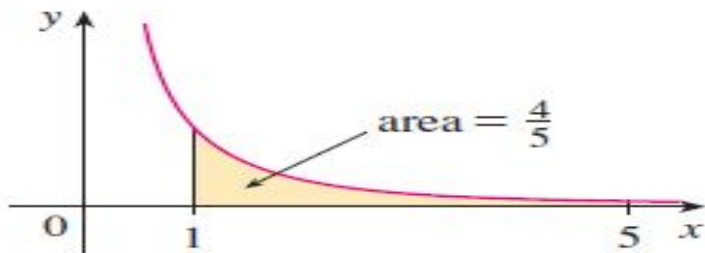
$$A(t) = \int_1^t \frac{1}{x^2} dx = 1 - \frac{1}{t}.$$



Improper integrals of type I: Infinite limits of integration

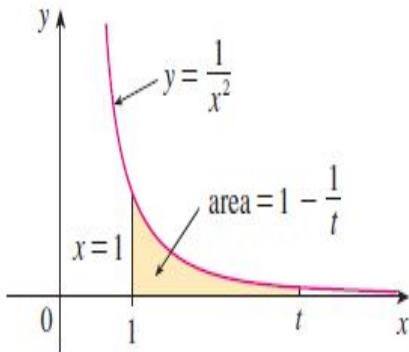


Improper integrals of type I: Infinite limits of integration



Improper integrals of type I: Infinite limits of integration

Observe that $A(t) < 1$ and $\lim_{t \rightarrow \infty} A(t) = \lim_{t \rightarrow \infty} (1 - \frac{1}{t}) = 1$.



So, we say that the **area of the infinite region S is equal to 1** and we write:

$$\int_1^{\infty} \frac{1}{x^2} dx = \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x^2} dx = 1.$$

Improper integrals of type I: Infinite limits of integration

Definition

(a) Suppose $\int_a^t f(x)dx$ exists for every number $t \geq a$. Then the **improper integral** of f over $[a, \infty)$ is defined by

$$\int_a^{\infty} f(x)dx = \lim_{t \rightarrow \infty} \int_a^t f(x)dx,$$

provided this limit exists (as a finite number).

Improper integrals of type I: Infinite limits of integration

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$$\int_a^\infty f(x)dx = \lim_{t \rightarrow \infty} \int_a^t f(x)dx,$$

provided this limit exists (as a finite number).

(b) Suppose $\int_t^a f(x)dx$ exists for every number $t \leq a$. Then

$$\int_{-\infty}^a f(x)dx = \lim_{t \rightarrow -\infty} \int_t^a f(x)dx$$

provided this limit exists (as a finite number).

Improper integrals of type I: Infinite limits of integration

Improper integrals of type I: Infinite limits of integration

- When the limit exists, we say that the improper integral **converges**; when the limit does not exist, we say that the improper integral **diverges**.

Definition

(c) If both $\int_a^\infty f(x)dx$ and $\int_{-\infty}^a f(x)dx$ are convergent, then we define:

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^{\infty} f(x)dx.$$

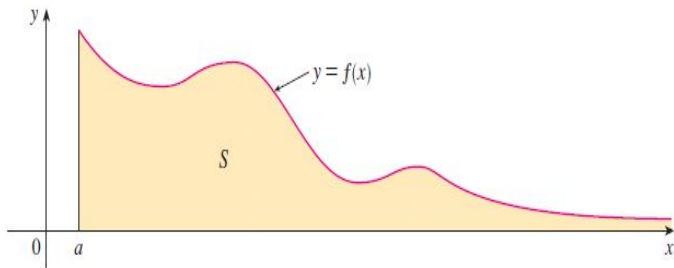
Here, any real number a can be used.

Improper integrals of type I: Infinite limits of integration

- Any of the improper integrals in Definition 7.1 can be interpreted as an area provided f is a positive function.

For example, if $f(x) \geq 0, x \in [a, \infty)$ and $\int_a^\infty f(x)dx$ exists then

$$\int_a^\infty f(x)dx = \text{the area of } S.$$



Improper integrals of type I: Infinite limits of integration

Example: Determine whether the following integral is convergent or divergent

$$a) \int_1^{\infty} \frac{1}{x} dx, \quad b) \int_1^{\infty} \frac{1}{x^2} dx.$$

Improper integrals of type I: Infinite limits of integration

Example: Determine whether the following integral is convergent or divergent

$$a) \int_1^{\infty} \frac{1}{x} dx, \quad b) \int_1^{\infty} \frac{1}{x^2} dx.$$

Solution: a) By definition

$$\int_1^{\infty} \frac{1}{x} dx = \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x} dx$$

Improper integrals of type I: Infinite limits of integration

Example: Determine whether the following integral is convergent or divergent

$$a) \int_1^{\infty} \frac{1}{x} dx, \quad b) \int_1^{\infty} \frac{1}{x^2} dx.$$

Solution: a) By definition

$$\int_1^{\infty} \frac{1}{x} dx = \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x} dx = \lim_{t \rightarrow \infty} (\ln x \Big|_1^t) = \lim_{t \rightarrow \infty} \ln t = \infty.$$

So the improper integral $\int_1^{\infty} \frac{1}{x} dx$ **diverges**.

Improper integrals of type I: Infinite limits of integration

Example: Determine whether the following integral is convergent or divergent

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Solution: a) By definition

$$\int_1^{\infty} \frac{1}{x} dx = \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x} dx = \lim_{t \rightarrow \infty} \left(\ln x \Big|_1^t \right) = \lim_{t \rightarrow \infty} \ln t = \infty.$$

So the improper integral $\int_1^{\infty} \frac{1}{x} dx$ **diverges**.

b) By definition

$$\int_1^{\infty} \frac{1}{x^2} dx = \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x^2} dx = \lim_{t \rightarrow \infty} \left(-\frac{1}{x} \Big|_1^t \right) = \lim_{t \rightarrow \infty} \left(-\frac{1}{t} + 1 \right) = 1.$$

So the improper integral $\int_1^{\infty} \frac{1}{x^2} dx$ **converges**.

Improper integrals of type I: Infinite limits of integration

Example:

$$c) \int_1^{\infty} \frac{1}{\sqrt{x}} dx.$$

Solution: c) By definition

$$\int_1^{\infty} \frac{1}{\sqrt{x}} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{1}{\sqrt{x}} dx$$

Improper integrals of type I: Infinite limits of integration

Example:

$$c) \int_1^{\infty} \frac{1}{\sqrt{x}} dx.$$

Solution: c) By definition

$$\int_1^{\infty} \frac{1}{\sqrt{x}} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{1}{\sqrt{x}} dx = \lim_{b \rightarrow \infty} \left(2\sqrt{x} \Big|_1^b \right) = 2 \lim_{b \rightarrow \infty} (\sqrt{b} - 1) = \infty.$$

So the improper integral **diverges**.

Theorem: For any $a > 0$, the improper integral

$$\int_a^\infty \frac{1}{x^p} dx \quad \text{is convergent if } p > 1 \quad \text{and divergent if } p \leq 1.$$

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Proof: If $p \neq 1$ then

$$\int_a^\infty \frac{1}{x^p} dx = \lim_{t \rightarrow \infty} \int_a^t \frac{1}{x^p} dx$$

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Proof: If $p \neq 1$ then

$$\begin{aligned} \int_a^\infty \frac{1}{x^p} dx &= \lim_{t \rightarrow \infty} \int_a^t \frac{1}{x^p} dx = \lim_{t \rightarrow \infty} \left(\frac{1}{1-p} x^{1-p} \right) \Big|_a^t \\ &= \frac{1}{1-p} \lim_{t \rightarrow \infty} t^{1-p} - \frac{1}{1-p} a^{1-p}. \end{aligned}$$

Theorem: For any $a > 0$, the improper integral

$$\int_a^{\infty} \frac{1}{x^p} dx \text{ is convergent if } p > 1 \text{ and divergent if } p \leq 1.$$

Proof: If $p \neq 1$ then

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Thus

$$\int_a^{\infty} \frac{1}{x^p} dx = -\frac{1}{1-p} a^{1-p} \text{ if } p > 1 \quad \int_a^{\infty} \frac{1}{x^p} dx = \infty \text{ if } p < 1.$$

Therefore, the improper integral $\int_a^{\infty} \frac{1}{x^p} dx$ is convergent if $p > 1$ and divergent if $p < 1$.

Theorem: For any $a > 0$, the improper integral

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Thus

$$\int_a^{\infty} \frac{1}{x^p} dx = -\frac{1}{1-p} a^{1-p} \text{ if } p > 1 \quad \int_a^{\infty} \frac{1}{x^p} dx = \infty \text{ if } p < 1.$$

Therefore, the improper integral $\int_a^{\infty} \frac{1}{x^p} dx$ is convergent if $p > 1$ and divergent if $p < 1$.

If $p = 1$ then

$$\int_a^{\infty} \frac{1}{x^p} dx = \int_a^{\infty} \frac{1}{x} dx = \lim_{t \rightarrow \infty} \int_a^t \frac{1}{x} dx = \lim_{t \rightarrow \infty} \ln x \Big|_a^t = \lim_{t \rightarrow \infty} (\ln t) - \ln a = \infty.$$

Therefore, the improper integral $\int_a^{\infty} \frac{1}{x} dx$ is divergent.

Improper integrals of type I: Infinite limits of integration

Example: Determine whether the following integral is convergent or divergent

$$a) \int_1^{\infty} e^{-x} dx, \quad b) \int_{-\infty}^{\infty} x e^{-x^2} dx.$$

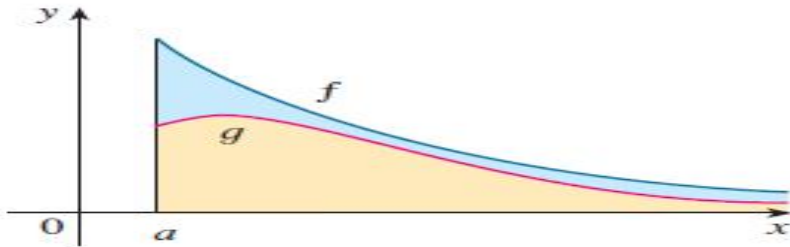
$$c) \int_{-\infty}^0 \frac{1}{\sqrt{3-x}} dx, \quad d) \int_1^{\infty} e^{-x^2} dx.$$

Improper integrals of type I: Comparison Theorem

Comparison Theorem Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

(a) If $\int_a^{\infty} f(x) dx$ is convergent, then $\int_a^{\infty} g(x) dx$ is convergent.

(b) If $\int_a^{\infty} g(x) dx$ is divergent, then $\int_a^{\infty} f(x) dx$ is divergent.



Improper integrals of type I: Comparison Theorem

Example: Determine whether the following integral is convergent or divergent

$$\int_0^{\infty} \frac{1}{1+e^x} dx.$$

Improper integrals of type I: Comparison Theorem

Example: Determine whether the following integral is convergent or divergent

$$\int_0^{\infty} \frac{1}{1+e^x} dx.$$

Solution: It is easy to see that

$$0 \leq \frac{1}{1+e^x} \leq \frac{1}{e^x}, \quad \forall x \geq 0.$$

Moreover, we have

$$\int_0^{\infty} \frac{1}{e^x} dx = \lim_{t \rightarrow \infty} -e^{-x} \Big|_0^t = \lim_{t \rightarrow \infty} (-e^{-t} + 1) = 1.$$

Since the improper integral $\int_0^{\infty} \frac{1}{e^x} dx$ is **convergent**, $\int_0^{\infty} \frac{1}{1+e^x} dx$ is **convergent**, **by the comparison theorem**.

Improper integrals of type I: Comparison Theorem

Example: Determine whether the following integral is convergent or divergent

$$a) \int_1^{\infty} \frac{\cos^2 x}{1+x^2} dx$$

$$b) \int_1^{\infty} \frac{\sqrt{x+x^2}}{x^2} dx.$$

Improper integrals of type I: Comparison Theorem

Example: Determine whether the following integral is convergent or divergent

$$a) \int_1^{\infty} \frac{\cos^2 x}{1+x^2} dx \qquad b) \int_1^{\infty} \frac{\sqrt{x+x^2}}{x^2} dx.$$

Solution: a) It is easy to see that

$$0 \leq \frac{\cos^2 x}{1+x^2} \leq \frac{1}{x^2}, \quad \text{for all } x \in [1, \infty).$$

Moreover, the improper integral $\int_1^{\infty} \frac{1}{x^2} dx$ **converges**. Thus $\int_1^{\infty} \frac{\cos^2 x}{1+x^2} dx$ **converges**, by **the comparison theorem**.

Improper integrals of type I: Comparison Theorem

Example: Determine whether the following integral is convergent or divergent

$$a) \int_1^{\infty} \frac{\cos^2 x}{1+x^2} dx \qquad b) \int_1^{\infty} \frac{\sqrt{x+x^2}}{x^2} dx.$$

Solution: a) It is easy to see that

$$0 \leq \frac{\cos^2 x}{1+x^2} \leq \frac{1}{x^2}, \quad \text{for all } x \in [1, \infty).$$

Moreover, the improper integral $\int_1^{\infty} \frac{1}{x^2} dx$ **converges**. Thus $\int_1^{\infty} \frac{\cos^2 x}{1+x^2} dx$ **converges**, by **the comparison theorem**.

b) It is clear that

$$\frac{\sqrt{x+x^2}}{x^2} \geq \frac{\sqrt{x^2}}{x^2} = \frac{1}{x} \geq 0, \quad \text{for all } x \in [1, \infty).$$

Moreover, the improper integral $\int_1^{\infty} \frac{1}{x} dx$ **diverges**. So $\int_1^{\infty} \frac{\sqrt{x+x^2}}{x^2} dx$ **diverges**, by **the comparison theorem**

Improper integrals of type II: Discontinuous integrands

Definition:

(a) If f is continuous on $[a, b)$ and is discontinuous at b , then

$$\int_a^b f(x)dx = \lim_{t \rightarrow b^-} \int_a^t f(x)dx$$

if this limit exists (as a finite number).

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if this limit exists (as a finite number).

(b) If f is continuous on $(a, b]$ and is discontinuous at a , then

$$\int_a^b f(x)dx = \lim_{t \rightarrow a^+} \int_t^b f(x)dx$$

if this limit exists (as a finite number).

Improper integrals of type II: Discontinuous integrands

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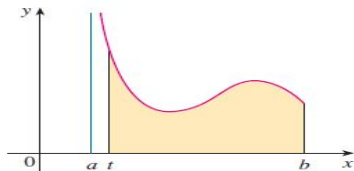
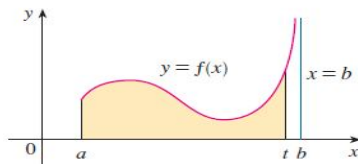
- The improper integral $\int_a^b f(x)dx$ is called **convergent** if the corresponding limit exists.
- If the limit does not exist, the integral is said to **diverge**.

(c) If f has a discontinuity at c , where $a < c < b$, and both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent, then we define:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Improper integrals of type II: Discontinuous integrands

$$\int_a^b f(x) dx = \lim_{t \rightarrow b^-} \int_a^t f(x) dx$$



$$\int_a^b f(x) dx = \lim_{t \rightarrow a^+} \int_t^b f(x) dx$$

Discontinuous integrands

Example: Evaluate the following integral or shows that it diverges.

$$\int_0^1 \frac{1}{x} dx.$$

Discontinuous integrands

Example: Evaluate the following integral or shows that it diverges.

$$\int_0^1 \frac{1}{x} dx.$$

Solution: Note that the integrand $f(x) = \frac{1}{x}$ is continuous on $[a, 1]$ for any $0 < a < 1$ and discontinuous at $x = 0$. By definition,

$$\int_0^1 \frac{1}{x} dx = \lim_{a \rightarrow 0^+} \int_a^1 \frac{1}{x} dx.$$

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$$\int_0^1 \frac{1}{x} dx.$$

Solution: Note that the integrand $f(x) = \frac{1}{x}$ is **continuous** on $[a, 1]$ for any $0 < a < 1$ and **discontinuous at $x = 0$** . By definition,

$$\int_0^1 \frac{1}{x} dx = \lim_{a \rightarrow 0^+} \int_a^1 \frac{1}{x} dx.$$

We have

$$\int_a^1 \frac{1}{x} dx = \ln x \Big|_a^1 = -\ln a.$$

Therefore,

$$\int_0^1 \frac{1}{x} dx = \lim_{a \rightarrow 0^+} \int_a^1 \frac{1}{x} dx = \lim_{a \rightarrow 0^+} (-\ln a) = \infty.$$

So the given integral diverges.

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We have

$$\int_a^1 \frac{1}{x^p} dx = \frac{1}{1-p} x^{1-p} \Big|_a^1 = \frac{1}{1-p} (1 - a^{1-p}).$$

Therefore,

$$\int_0^1 \frac{1}{x^p} dx = \lim_{a \rightarrow 0^+} \int_a^1 \frac{1}{x^p} dx = \lim_{a \rightarrow 0^+} \left(\frac{1}{1-p} (1 - \frac{1}{a^{p-1}}) \right) = \infty.$$

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Therefore,

$$\int_0^1 \frac{1}{x^p} dx = \lim_{a \rightarrow 0^+} \int_a^1 \frac{1}{x^p} dx = \lim_{a \rightarrow 0^+} \left(\frac{1}{1-p} (1 - a^{1-p}) \right) = \frac{1}{1-p}.$$

Improper integrals of type II: Discontinuous integrands

In summary, we have

Theorem The improper integral

$$\int_0^1 \frac{1}{x^p} dx$$

is convergent if $p < 1$ and divergent if $p \geq 1$.

Improper integrals of type II: Discontinuous integrands

Example: Evaluate the following integral or shows that it diverges.

$$\int_0^1 \ln x dx$$

Improper integrals of type II: Discontinuous integrands

Example: Evaluate the following integral or shows that it diverges.

$$\int_0^1 \ln x dx$$

Solution: Note that the integrand $f(x) = \ln x$ is continuous on $[\epsilon, 1]$ for any $0 < \epsilon < 1$ and discontinuous at $x = 0$. By definition,

$$\int_0^1 \ln x dx = \lim_{\epsilon \rightarrow 0^+} \int_{\epsilon}^1 \ln x dx.$$

Improper integrals of type II: Discontinuous integrands

Example: Evaluate the following integral or shows that it diverges.

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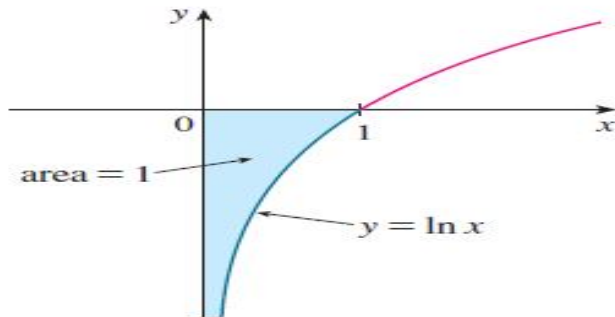
Using integration by part, we have

$$\int_{\epsilon}^1 \ln x dx = x \ln x \Big|_{\epsilon}^1 - \int_{\epsilon}^1 x \frac{1}{x} dx = -\epsilon \ln \epsilon - (1 - \epsilon).$$

Thus

$$\int_0^1 \ln x dx = \lim_{\epsilon \rightarrow 0^+} (-\epsilon \ln \epsilon - (1 - \epsilon)) = -1.$$

Improper integrals of type II: Discontinuous integrands



Example

Evaluate the following integral or shows that it diverges.

$$\int_0^3 \frac{1}{\sqrt{3-x}} dx.$$

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$$\int_0^3 \frac{1}{\sqrt{3-x}} dx.$$

Solution: Note that the integrand $f(x) = \frac{1}{\sqrt{3-x}}$ is continuous on $[0, t]$ for any $0 < t < 3$ and discontinuous at $x = 3$. By definition,

$$\int_0^3 \frac{1}{\sqrt{3-x}} dx = \lim_{t \rightarrow 3^-} \int_0^t \frac{1}{\sqrt{3-x}} dx.$$

Example

Evaluate the following integral or shows that it diverges.

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$$\int_0^3 \frac{1}{\sqrt{3-x}} dx = \lim_{t \rightarrow 3^-} \int_0^t \frac{1}{\sqrt{3-x}} dx.$$

We have

$$\int_0^t \frac{1}{\sqrt{3-x}} dx = -2\sqrt{3-x} \Big|_0^t = -2\sqrt{3-t} + 2\sqrt{3}.$$

Therefore,

$$\int_0^3 \frac{1}{\sqrt{3-x}} dx = \lim_{t \rightarrow 3^-} \int_0^t \frac{1}{\sqrt{3-x}} dx = \lim_{t \rightarrow 3^-} (-2\sqrt{3-t} + 2\sqrt{3}) = 2\sqrt{3}.$$

Example: Evaluate the following integral or shows that it diverges.

$$\int_0^2 \frac{1}{1-x} dx.$$

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$$\int_0^2 \frac{1}{1-x} dx = \int_0^1 \frac{1}{1-x} dx + \int_1^2 \frac{1}{1-x} dx.$$

Here $\int_0^1 \frac{1}{1-x} dx$ and $\int_1^2 \frac{1}{1-x} dx$ are improper integrals.

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Here $\int_0^1 \frac{1}{1-x} dx$ and $\int_1^2 \frac{1}{1-x} dx$ are improper integrals. Since

$$\int_0^1 \frac{1}{1-x} dx = \lim_{b \rightarrow 1^-} \int_0^b \frac{1}{1-x} dx = \lim_{b \rightarrow 1^-} (-\ln |1-t|) = \infty.$$

So the integral

$$\int_0^2 \frac{1}{1-x} dx,$$

diverges.

4.7 IMPROPER INTEGRALS

4.7.3 COMPARISON OF INTEGRALS

Theorem (Comparison Test for Improper Integrals)

Suppose that functions f and g are continuous on $(a, b]$ and satisfy

$$0 \leq f(x) \leq g(x), \quad x \in (a, b].$$

(i) If $\int_a^b g(x)dx$ converges, then so does $\int_a^b f(x)dx$, and

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx.$$

(ii) If $\int_a^b f(x)dx$ diverges, then $\int_a^b g(x)dx$ also diverges.

4.7 IMPROPER INTEGRALS

4.7.3 COMPARISON OF INTEGRALS

Example: Show that $\int_0^1 \frac{e^{-x^2}}{x^p} dx$ converges for any $0 < p < 1$

Solution: We have

$$0 \leq \frac{e^{-x^2}}{x^p} \leq \frac{1}{x^p}, \quad \forall x \in (0, 1].$$

Moreover, $\int_0^1 \frac{1}{x^p} dx$ converges for any $0 < p < 1$. By the comparison theorem, $\int_0^1 \frac{e^{-x^2}}{x^p} dx$ converges.

Example: Determine whether $\int_0^{\pi/4} \frac{dx}{x^4 \cos x}$ converges or diverges.

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(i) If $\int_a^b g(x)dx$ converges, then so does $\int_a^b f(x)dx$, and

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx.$$

(ii) If $\int_a^b f(x)dx$ diverges, then $\int_a^b g(x)dx$ also diverges.

Example: Determine whether $\int_0^1 \frac{(2+\cos x)dx}{1-x}$ converges or diverges.